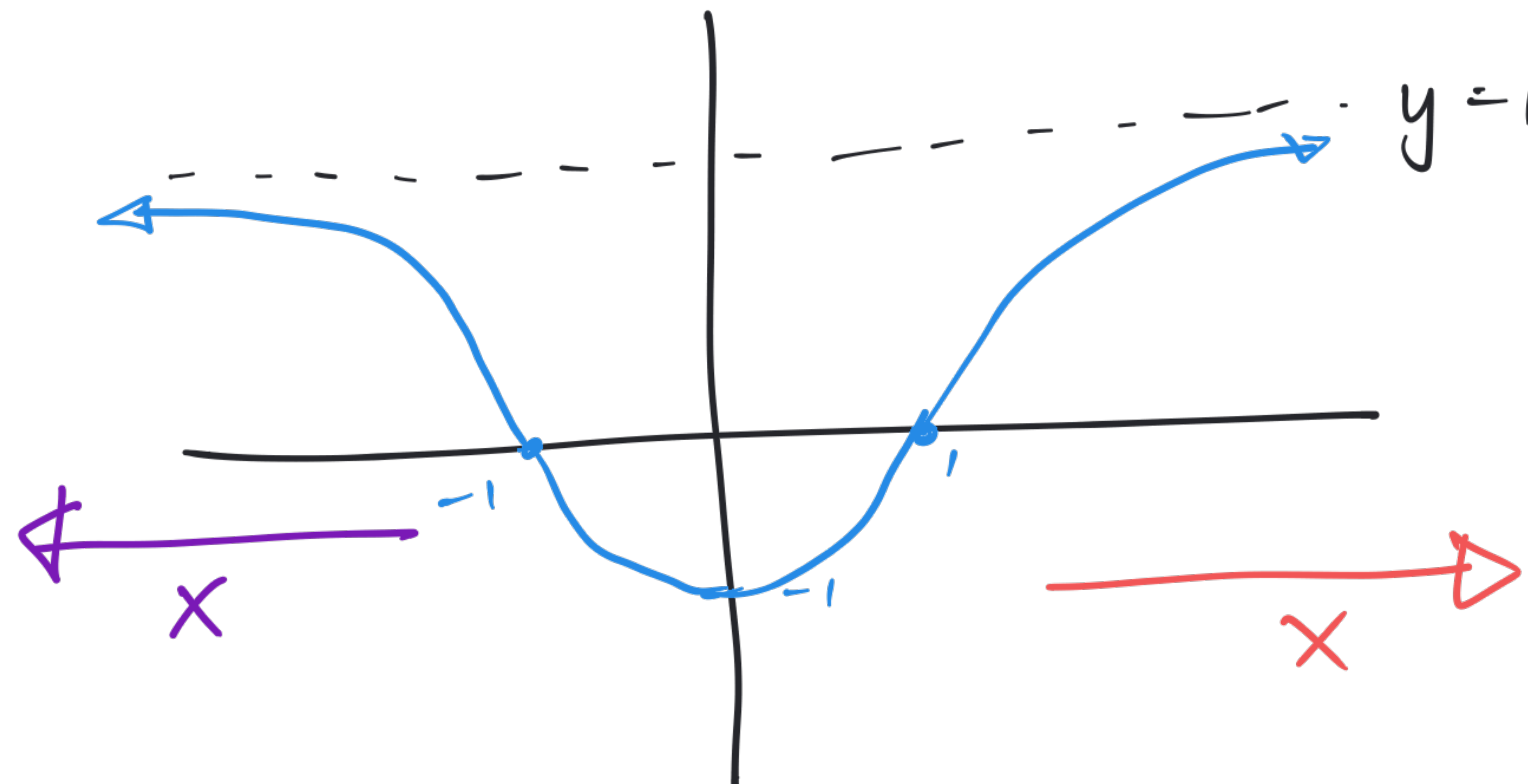
Intro Video: section 2.6
limits at infinity

Math F251X: Calculus 1

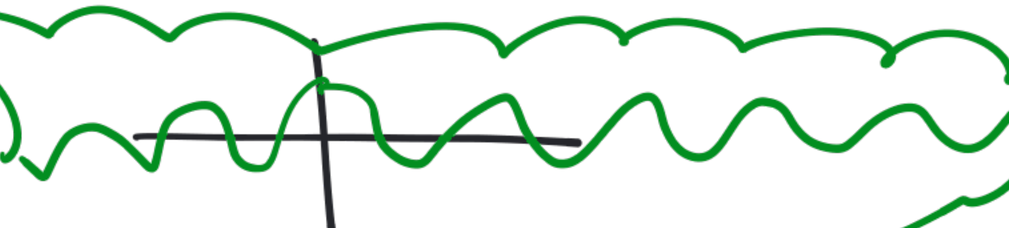
Our question today: As x gets really big, what happens to $f(x)$?

- What is $\lim_{x \to \infty} f(x)$?
- What is $\lim_{x \to -\infty} f(x)$?



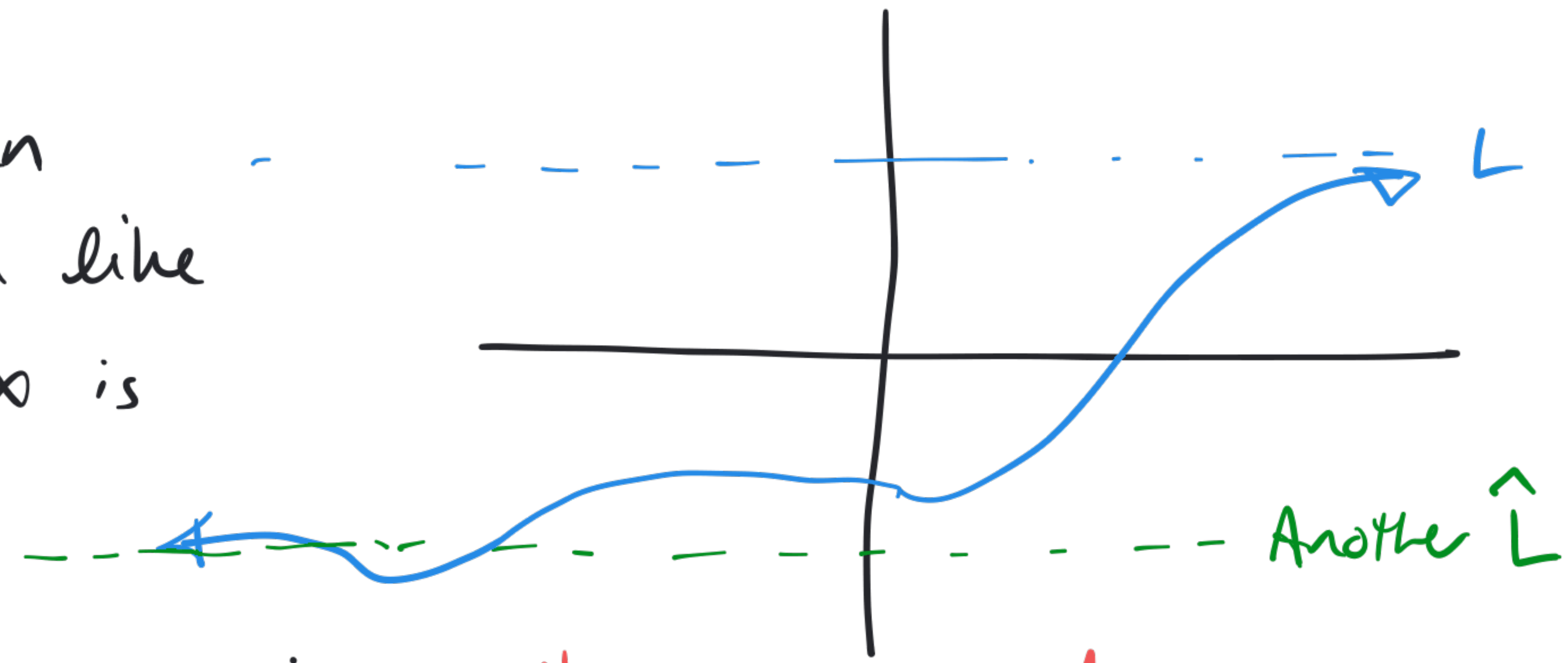
THREE POSSIBILITIES:

1. As $x \to \infty$, $f(x) \to \infty$ or $-\infty$
   $x \to -\infty$
   Polynomials!
2. As $x \to \infty$, $f(x) \to L$ where L is some fixed (finite) #
   $x \to -\infty$
   $f(x) = \arctan(x)$
3. As $x \to \infty$, $f(x)$ does neither.
   $x \to -\infty$
   $f(x) = \cos(x)$



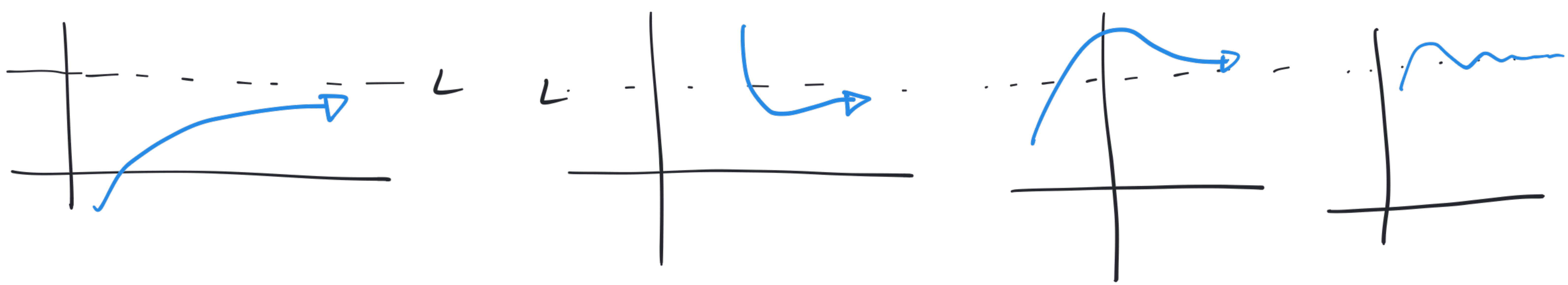
Suppose $\lim_{x \to \infty} f(x) = L$

This means, $f(x)$ can get as close as you like to L, as long as x is large enough!



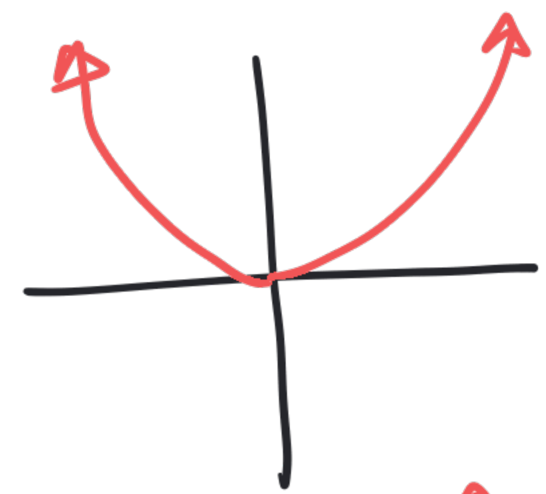
$\longrightarrow$ The line $y = L$ is a HORIZONTAL ASYMPTOTE

$\boxed{\text{Asymptotic behavior:}}$



Examples we know:

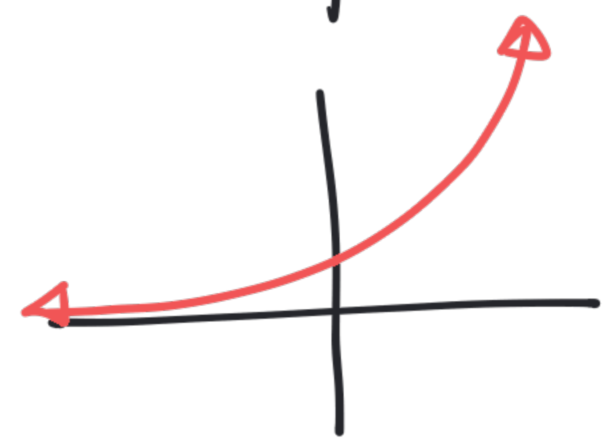
1. $f(x) = x^2$



$$\lim_{x \to \infty} x^2 = \infty$$

$$\lim_{x \to -\infty} x^2 = \infty$$

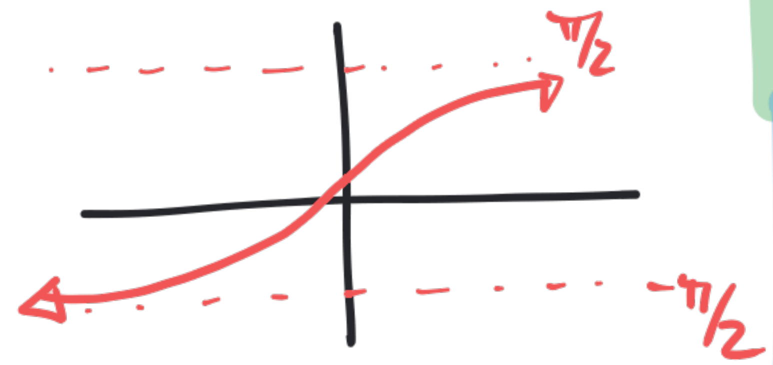
2. $g(x) = e^x$



$$\lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to -\infty} e^x = 0$$ ← $y=0$ is a HA

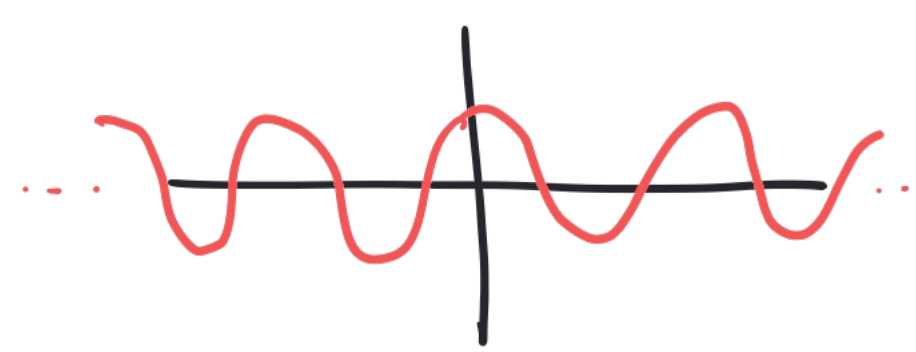
3. $h(x) = \arctan(x)$



$$\lim_{x \to \infty} \arctan(x) = \pi/2$$

$$\lim_{x \to -\infty} \arctan(x) = -\pi/2$$

4. $j(x) = \cos(x)$



$\lim_{x \to \infty} \cos(x)$ DNE

$\lim_{x \to -\infty} \cos(x)$ DNE

Determining limits at ∞ for rational functions:

rational function = quotient of polynomials = $\dfrac{\text{polynomial}}{\text{polynomial}}$

Trick: divide top and bottom by the highest power of x in the denominator

Highest power of x in denominator is 1

Example Determine $\displaystyle\lim_{x\to\infty}\left(\frac{2x+5}{x-4}\right)\left(\frac{1/x}{1/x}\right)$

$\dfrac{5}{1000000000}$

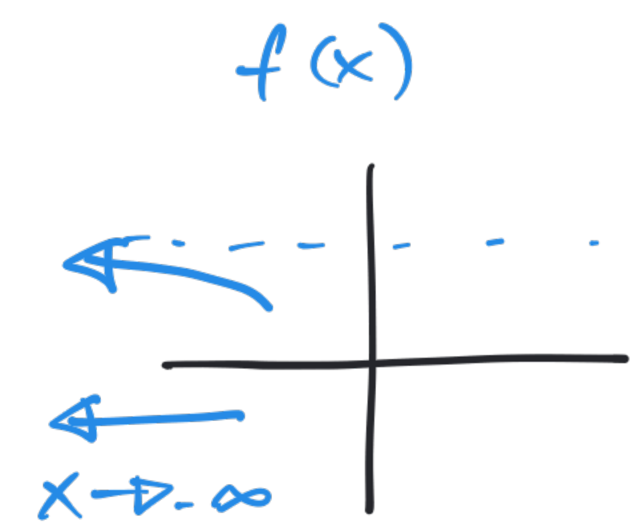
$$= \lim_{x\to\infty}\frac{2+5/x}{1-4/x} = \frac{\displaystyle\lim_{x\to\infty} 2+5/x}{\displaystyle\lim_{x\to\infty} 1-4/x} = \frac{\displaystyle\lim_{x\to\infty} 2 + \lim_{x\to\infty} 5/x}{\displaystyle\lim_{x\to\infty} 1 - \lim_{x\to\infty} 4/x} = \frac{2+0}{1-0} = 2$$

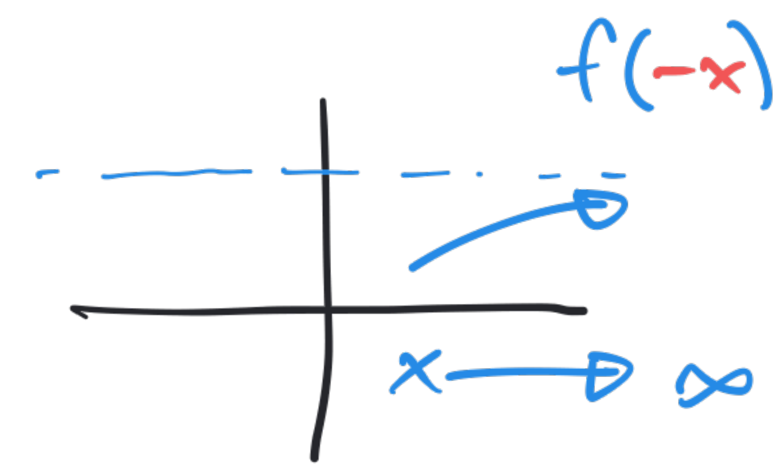
Example: $\displaystyle\lim_{x\to\infty}\left(\frac{x+4}{x^2+x-3}\right)\left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty}\frac{1/x + 4/x^2}{1+1/x-3/x^2} = \frac{0+0}{1+0+0} = 0.$

$\dfrac{x+4}{x^2+x-3}$ "behaves like" $\dfrac{x}{x^2} = \dfrac{1}{x}$ and $\dfrac{2x+5}{x-4}$ "behaves" like $\dfrac{2x}{x}$

What about limits as $x \to -\infty$?

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x)$$





Example $\displaystyle\lim_{x\to-\infty}\frac{2x+5}{x-4}$

$$= \lim_{x\to\infty}\frac{2(-x)+5}{-x-4} = \lim_{x\to\infty}\left(\frac{-2x+5}{-x-4}\right)\left(\frac{1/x}{1/x}\right)$$

$$= \lim_{x\to\infty}\frac{-2+5/x}{-1-4/x} = \frac{-2+0}{-1-0} = 2.$$

Example: $\displaystyle\lim_{x\to-\infty}\frac{2x^3+5}{3x^2+1}$

"Looks like" $\frac{2x^3}{3x^2}$ guess blows up to $-\infty$

$$= \lim_{x\to\infty}\frac{2(-x)^3+5}{3(-x)^2+1} = \lim_{x\to\infty}\left(\frac{-2x^3+5}{3x^2+1}\right)\left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty}\frac{\overset{\to -\infty}{-2x+5/x^2}}{\underset{\to 3}{3+1/x^2}}$$

$$= -\infty$$

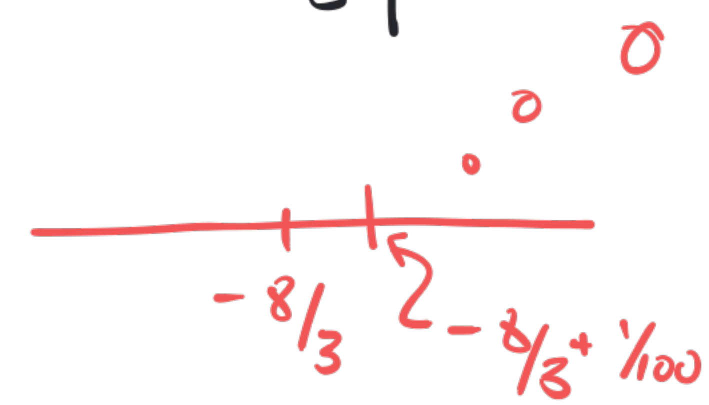
Example: Determine all horizontal and vertical asymptotes of the function $f(x) = \dfrac{2x-4}{3x+8}$

① Domain of $f(x)$ is $\{x \in \mathbb{R} : x \neq \frac{-8}{3}\}$

② $$\lim_{x \to -8/3^{+}} \frac{2x-4}{3x+8} = -\infty$$

$2x-4 \to 2(-8/3)-4 < 0$

$3x+8 \to 0^{+}$



$3\left(\frac{-8}{3}+\frac{1}{100}\right)+8$

$= -8+\frac{1}{100}+8$

>0

(and $\displaystyle\lim_{x \to -\frac{8}{3}^{-}} \frac{2x-4}{3x+8} = \infty$)

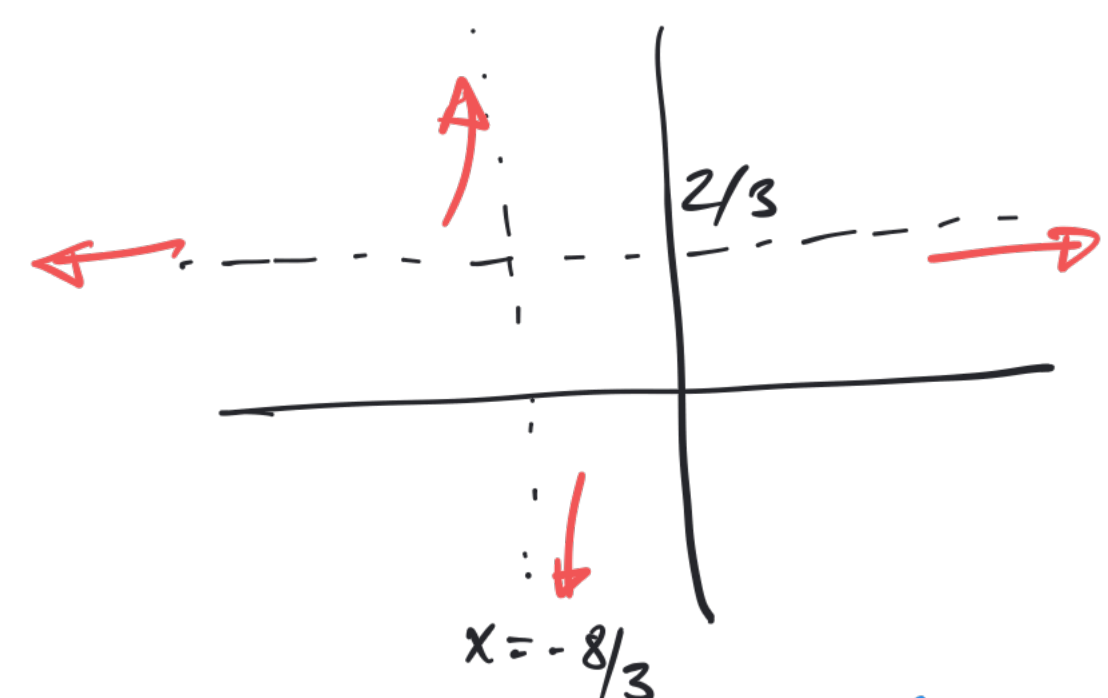
$$\boxed{x = \frac{-8}{3} \text{ is a VA}}$$

② $$\lim_{x \to \infty} \left(\frac{2x-4}{3x+8}\right)\frac{1/x}{1/x} = \lim_{x \to \infty} \frac{2-4/x}{3+8/x} = \frac{2}{3}$$

$$\lim_{x \to -\infty} \frac{2x-4}{3x+8} = \lim_{x \to \infty} \frac{2(-x)-4}{3(-x)+8}$$

$$= \lim_{x \to \infty} \left(\frac{-2x-4}{-3x+8}\right)\frac{1/x}{1/x}$$

$$= \lim_{x \to \infty} \frac{-2-4/x}{-3+8/x} = \frac{2}{3}$$



$$\boxed{y = 2/3 \text{ is a HA}}$$

Still to come...

- more complicated limits

  "type" $\infty - \infty$
  "type" $0/0$
  } Need more algebraic tricks!

- Squeeze Theorem